

# THE CONSTRUCTION OF SORKIN TRIANGULATIONS

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Some time ago, Sorkin (1975) reported investigations of the time evolution and initial value problems in Regge calculus, for one triangulation each of the manifolds  $\mathbf{R} \times \mathbf{S}^3$  and  $\mathbf{R}^4$ . Here we display the simple, local characteristic of those triangulations which underlies the structure found by Sorkin, and emphasise its general applicability, and therefore the general validity of Sorkin's conclusions. We also make some elementary observations on the resulting structure of the time evolution and initial value problems in Regge calculus, and add some comments and speculations.

In Regge calculus (Regge 1961, see Williams and Tuckey 1992 for a brief review), the 4-dimensional spacetime manifold  $M$  is triangulated, i.e. divided into cells which are 4-simplices, and the metric is approximated as being flat on the interiors of these cells. The metric is determined by the squared lengths  $s_\lambda$  of the 1-simplices ("edges")  $\lambda$  in the triangulation, and curvature is concentrated on the 2-simplices ("bones"). The Einstein-Hilbert action  $\int R\sqrt{|g|}d^4x$  can be evaluated for these piecewise-flat metrics, and leads to

$$S = \sum_{\text{bones } b} \varepsilon_b A_b , \quad (1)$$

where  $A_b$  is the area of bone  $b$ , and  $\varepsilon_b$  is its deficit angle, defined as follows.

Let  $\alpha$  be a  $(0, 1, 2, 3$  or  $4)$ -simplex. Define the " $n$ -star of  $\alpha$ ",  $\mathcal{S}_n(\alpha)$ , to be the set of all  $n$ -simplices which either contain or are contained in  $\alpha$ . (The extension to the  $n$ -star of a set of simplices is straightforward.) The deficit angle of bone  $b$  is

$$\varepsilon_b = 2\pi - \sum_{c \in \mathcal{S}_4(b)} \theta_{bc} , \quad (2)$$

where  $\theta_{bc}$  is the dihedral angle between the two tetrahedral faces of cell  $c$  which meet at bone  $b$ , measured internally to  $c$ .

The discrete "equations of motion", approximating Einstein's equations, are the Regge equations,

$$\frac{\partial S}{\partial s_\lambda} = 0 \quad \forall \text{ edges } \lambda , \quad (3)$$

i.e. one algebraic equation for each edge  $\lambda$ . It was proved by Regge (1961) and Sorkin (1975) that the terms resulting from the variations of the deficit angles in (1) cancel, so equations (3) become just

$$\sum_{\text{bones } b} \varepsilon_b \frac{\partial A_b}{\partial s_\lambda} = 0 \quad \forall \lambda . \quad (4)$$

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It follows that the equation of motion for edge  $\lambda$  involves only  $\mathcal{S}_4(\lambda)$  (Sorkin 1975). Following Sorkin, if edge  $\mu \in \mathcal{S}_1(S_4(\lambda))$ , we say that “ $\lambda$  implicates  $\mu$ ”.

In the two cases he reported, Sorkin (1975) found that for the time evolution problem the Regge equations separated into small sets, which could be solved independently to evolve a local region of a spacelike hypersurface forward in time. This structure is much simpler than in the “3+1” and “null-strut” approaches developed later (Miller 1986, Porter 1987, Tuckey 1989), where the evolution equations across a whole hypersurface are coupled, giving a massive algebraic problem to solve at each step in a time evolution calculation. We believe that the reason that Sorkin’s results have not attracted more attention is that the simple property of his triangulations which underlies them was not generally understood, so the general validity of his findings was not appreciated. We hope to correct this situation with this note.

Sorkin also identified a sufficient amount of initial data needed to allow a time evolution calculation to commence, as well as the Regge equations which concern only these data, which together constitute the initial value problem in Regge calculus. Our emphasis on the basic structure of the triangulations used also allows an interesting further analysis of this problem.

### *Time Evolution Problem*

We assume  $M$  has topology  $\mathbf{R} \times \Sigma$ , where  $\Sigma$  is a 3-manifold, taken to have no boundary for convenience. To display the typical step in a time evolution calculation we assume we have initial data on a slice of  $M$ , which is bounded by 3-dimensional hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  having the topology of  $\Sigma$ , and which divides  $M$  into two separate regions, denoted as “past” and “future”. More specifically, we assume we have some triangulation of this 4-dimensional slice (which induces triangulations of  $\Sigma_1$  and  $\Sigma_2$ ), and that the squared lengths of all edges in this triangulated region are known. We further require that no edge should lie in both  $\Sigma_1$  and  $\Sigma_2$ . Thus  $\Sigma_1$  and  $\Sigma_2$  can intersect at vertices of their triangulations, but not on higher-dimensional simplices.

The typical time evolution step proceeds by extending the triangulation into the future region of  $M$ , from a local region of the future hypersurface, say  $\Sigma_2$ . This is done by choosing some vertex  $\lambda$  on  $\Sigma_2$ , adding a new vertex  $\lambda'$  in the future region of  $M$ , and then attaching a new 4-simplex to each tetrahedron in  $\Sigma_2$  which contains  $\lambda$ , with all of these new 4-simplices sharing  $\lambda'$  as their one vertex which is not in  $\Sigma_2$ . The figure shows the 3-dimensional analogue of this procedure.

Denote by  $\mu_1, \dots, \mu_{N_1}$  the elements of  $\mathcal{S}_0(S_1(\lambda))$  which lie in  $\Sigma_2$ , not including  $\lambda$ .  $N_1$  is the number of edges on  $\Sigma_2$  which intersect at  $\lambda$ . Our construction has introduced  $N_1 + 1$  new edges; the  $N_1$  “diagonal” edges  $\mu_1\lambda', \mu_2\lambda', \dots, \mu_{N_1}\lambda'$ , and the “vertical” edge  $\lambda\lambda'$ . We have also made  $N_1 + 1$  new edges lie in the interior of the triangulated region, the  $N_1$  “horizontal” edges  $\mu_1\lambda, \mu_2\lambda, \dots, \mu_{N_1}\lambda$ , as well as  $\lambda\lambda'$ . The equations of motion for these new interior edges implicate the  $N_1 + 1$  new edges, plus edges lying in the initial slice whose squared lengths are already known. Thus, in principle, we can solve these equations to find the squared lengths of the new edges.

We note that the extended, triangulated slice of  $M$  thus obtained satisfies the conditions required of the initial data, so the process may be repeated again at any vertex on the new future boundary hypersurface. Clearly this construction is purely local, and

may be carried out regardless of the topology of  $\Sigma$ , or the triangulation of  $\Sigma_2$ . Thus this construction allows the time evolution problem in Regge calculus to consist in general of the successive solution of sets of  $N_1 + 1$  equations in  $N_1 + 1$  unknowns, where  $N_1$  is the number of edges in the current hypersurface meeting on the relevant vertex.

We refer to this basic evolution step as “the evolution of a vertex”, and call a triangulation built up from a suitable initial slice by a series of such steps a “Sorkin triangulation”. We note that there is a one to one correspondence between the horizontal and diagonal edges referred to above, and indeed between all simplices in the original hypersurface  $\Sigma_2$  and those in the new future boundary of the triangulated region. In fact, the evolution of a vertex leaves invariant both the topology and triangulation of the spacelike hypersurfaces. Thus a Sorkin triangulation may be viewed as being based on a sequence of identically triangulated 3-dimensional hypersurfaces, which overlap each other in some regions and which are separated by vertical edges (and associated 4-simplices) in others.

Presumably one would work systematically across a hypersurface, evolving all vertices once before any is evolved twice, although one might choose not to evolve some region to avoid a singularity, for example. It is worth noting that any two vertices on a hypersurface which are not connected by an edge may be evolved independently and hence simultaneously, as the corresponding sets of equations are not coupled. Thus the method lends itself to parallel processing.

For example, if  $\Sigma$  is  $S^3$  and the triangulation of a given hypersurface is the 600-tetrahedron regular triangulation, having 120 vertices and 12 edges meeting on each vertex, then all vertices could be evolved forward once by a process of 4 steps, in each of which 30 are evolved in parallel (Williams 1990). The evolution of each vertex would involve the solution of 13 equations in 13 unknowns.

### *Initial Value Problem*

The initial value problem, in short, is to ensure that the initial data, i.e. the squared lengths of the edges in the initial slice, satisfy the relevant Regge equations. The relevant equations are those for edges lying in the interior of the initial slice, since these implicate only edges in the initial slice. (The equations for edges lying on the boundary of the slice are relevant to the evolution of the metric into the future or past regions of  $M$ , as described above.)

The initial value problem may be viewed as a sequence of incomplete time evolution steps in the following way. We commence with the 3-dimensional hypersurface  $\Sigma_1$ , triangulated in some way, and we freely specify the squared lengths of the edges in this triangulation. Then we attempt to evolve a vertex on  $\Sigma_1$  in the way described above. This introduces only one new edge on the interior of the triangulated region, the vertical edge, so the  $N_1 + 1$  new edge lengths may be chosen subject only to the equation corresponding to this edge. (Of course, the simplex inequalities which ensure that the metric is Minkowskian on the interiors of the 4-simplices must also be respected.)

Many vertices on  $\Sigma_1$  may be evolved in this way, but eventually we reach a stage where the evolution of a vertex involves adding 4-simplices onto 4-simplices which have previously been added to  $\Sigma_1$ . At this stage we will have to satisfy both the equation for the new vertical edge, and those for any horizontal edges which are now put into the interior of the triangulated region. We continue making such partial evolution steps until any further

evolution would involve a full set of  $N_1 + 1$  equations, when the initial data is complete. Thus the initial value problem may be viewed as the solution of a sequence of sets of less than  $N_1 + 1$  equations in  $N_1 + 1$  variables.

Note that any further evolution step involves a full complement of equations when the initial slice has reached a state where no edge in the future boundary hypersurface also lies in  $\Sigma_1$ , which gives the condition we required of the initial data earlier. For example, taking the 600-tetrahedron triangulation of  $S^3$  mentioned above, this could be achieved by making 3 steps, in each of which 30 vertices are evolved, leaving 30 vertices on  $\Sigma_1$  unevolved. These considerations are consistent with the initial data identified by Sorkin (1975) in his examples, but the analysis given here is new.

### *Further Comments*

We have not quantitatively addressed the crucial question of the existence of solutions, but we speculate that relativistic causality is relevant here. We expect that in the evolution of any vertex  $\lambda$  the diagonal edges will typically be spacelike or null, since we expect the new 4-simplices added to lie within the future light cone of the known part of  $S_4(\lambda)$ , which contains all the initial data for the evolution step. The vertical edge may be timelike, null or spacelike, but the condition on the diagonal edges implies a restriction on this edge, corresponding to a Courant limit. We thus expect the family of 3-dimensional hypersurfaces referred to above to be spacelike (or null) typically, while the edges going between them can be timelike, spacelike or null.

In making a time evolution calculation, it is thought (Tuckey 1991, Galassi and Miller 1992) that it will be appropriate to exploit the approximate symmetries of Regge calculus (Roček and Williams 1984, Hartle 1985, Piran and Strominger 1986) to allow some of the new, unknown edge lengths to be freely specified, and to ignore some of the equations of motion (or rather to use them as a check on the solutions). Important work on this point has been carried out by Galassi and Miller (1992), both in demonstrating numerically the effects of the approximate symmetries on the Regge equations, and in identifying combinations of edge lengths to treat as “gauge variables”. This approach corresponds to freely specifying the discretised lapse and shift in finite-difference approximations to Einstein’s equations, and using the corresponding constraint equations only to check the solutions, even though the diffeomorphism symmetry of the continuum theory is broken in this approximation. In the continuum limit the diffeomorphism symmetry should be restored (in both approximations), so in this limit the solutions will satisfy all equations.

This approach is in contrast with that envisaged by Sorkin, who wished to solve all equations at each time evolution step to find all new edge lengths, thus “allowing the equations to choose the gauge”. It is expected that this approach would lead to ill-conditioned algebraic systems, because of the approximate freedom in the solutions, as found in previous numerical work (Hartle 1986).

Barrett (1992) has given a beautiful interpretation of the basic evolution step described here as a sequence of elementary moves on the triangulation of the spacelike hypersurface, satisfying a “regularity condition”: that the number of new edges introduced is equal to the number of new edges on the interior of the triangulated region. From this point of view we may state an important question: are there sequences of elementary moves which satisfy the regularity condition, but which change the triangulation of the spacelike hypersurfaces?

Such sequences would provide a way of adaptively refining the triangulation of space, allowing the insertion or deletion of simplices as appropriate to follow the evolution of detail in the solution, implementing Barrett’s “toolkit”.

Many other important problems require further attention. Primary among these is the existence of solutions to the sets of equations described here, both in the time evolution and initial value problems. We also need to verify the presence of local causality in this method, as outlined above, and the corresponding existence of a Courant limit. The detailed effects of the approximate symmetry, and its optimum exploitation, also require investigation. These issues are being addressed in current work (Barrett *et al* 1993, Galassi and Miller 1993).

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## Figure caption

The 3-dimensional analogue of the basic evolution step described in the text. We assume no edges on surface  $\Sigma_2$  also lie on  $\Sigma_1$ . All tetrahedra shown here are new, added onto  $\Sigma_2$ . The “diagonal” edges  $\mu_1\lambda', \dots, \mu_{N_1}\lambda'$  and the “vertical” edge  $\lambda\lambda'$  are new. The “horizontal” edges  $\mu_1\lambda, \dots, \mu_{N_1}\lambda$  and the vertical edge lie on the interior of the triangulated region, and provide equations of motion to be solved for the squared lengths of the new edges.